

# Math 600 Day 7: Whitney Embedding Theorem

Ryan Blair

University of Pennsylvania

Thursday September 30, 2010

# Outline

- 1 Tangent Bundles and Derivatives of maps
- 2 Whitney Embedding Theorem

Let  $M$  be a  $k$ -dimensional differentiable manifold in  $\mathbb{R}^n$ , and let  $(U_1, f_1)$  be a coordinate system around  $x \in M$ .

Since  $f_1'(u_1)$  has rank  $k$ , the linear transformation  $(f_1)' : \mathbb{R}_{u_1}^k \rightarrow \mathbb{R}_x^n$  is one-to-one, and hence  $(f_1)'(\mathbb{R}_{u_1}^k)$  is a  $k$ -dimensional subspace of  $\mathbb{R}_x^n$ .

If  $(U_2, f_2)$  is another coordinate system around  $x \in M$ , then

$$(f_2)'(\mathbb{R}_{u_2}^k) = (f_1)'(f_1^{-1} \circ f_2)'(\mathbb{R}_{u_2}^k) = (f_1)'(\mathbb{R}_{u_1}^k).$$

Thus the  $k$ -dimensional subspace  $(f_1)'(\mathbb{R}_{u_1}^k)$  does not depend on the choice of coordinate system around  $x$ .

This subspace is denoted by  $M_x$  or  $TM_x$  or  $T_xM$ , and is called the **tangent space** to  $M$  at  $x$ .

## Definition

Let  $M^m \subset \mathbb{R}^N$  be a smooth manifold. The tangent spaces  $T_x M$  to  $M$  at its various points are all vector subspaces of  $\mathbb{R}^N$  (In an embedded picture the  $T_x M$  can overlap as  $x$  varies.)

The tangent bundle  $TM$  is defined to be the set

$$TM = \{(x, v) : x \in M, v \in T_x M\},$$

and is thus the disjoint union of all the tangent spaces  $T_x M$ .

A map  $g : M \rightarrow N$  between differentiable manifolds (concrete or abstract) is said to be differentiable at the point  $x \in M$  if there are coordinate charts  $(U, f)$  about  $x$  in  $M$  such that  $f(x) = u$  and  $(V, h)$  about  $y = g(x)$  in  $N$  and such that  $h^{-1} \circ g \circ f$ , suitably restricted, is differentiable at  $u$ .

The derivative  $g'(x) : T_x M \rightarrow T_y N$  is then defined to be the unique linear map such that

$$(h \circ g \circ f^{-1})'(u) = h'(y) \circ g'(x) \circ (f^{-1})'(u).$$

For concrete manifolds the tangent bundle  $TM$  is a subset of  $\mathbb{R}^N \times \mathbb{R}^N$ , and is easily seen to be a smooth manifold of dimension  $2m$ . If  $f : M \rightarrow N$  is a smooth map of smooth manifolds, then the map

$$df : TM \rightarrow TN,$$

defined by  $df(x, v) = (f(x), df_x(v))$ , is also a smooth map.

### Remark

The tangent bundle of a smooth manifold can also be defined abstractly, without reference to  $\mathbb{R}^N$ .

### Definition

If  $f : M \rightarrow N$  is a smooth map between smooth manifolds,  $f$  is an immersion at  $x$  if  $df_x : T_x M \rightarrow T_x N$  is injective.

### Definition

A map  $f : M \rightarrow N$  is proper if the preimage of every compact set in  $N$  is compact in  $M$ .

### Definition

An immersion that is injective and proper is called an embedding.

**Note:** Given an injective immersion  $f : M \rightarrow N$  such that  $M$  is compact, then  $f$  is automatically an embedding.

## Theorem

*Every smooth  $m$ -dimensional manifold admits a smooth embedding into Euclidean space  $\mathbb{R}^{2m+1}$ .*



# Proof of the Whitney Embedding Theorem.

We give the proof first for smooth manifolds already embedded in some large-dimensional Euclidean space, and at the end we sketch how to adjust the argument for abstract smooth manifolds.

**Step 1.** In this step we show that  $M^m$  admits a smooth, one-to-one immersion into  $\mathbb{R}^{2m+1}$ . If  $M$  is compact, this is same thing as an embedding, but if  $M$  is not compact, it may not be the same thing. For example, a line of irrational slope on the torus  $S^1 \times S^1$  is a smooth immersion of  $\mathbb{R}$  into the torus, but not an embedding.

We start out with  $M^m$  a subset of some large Euclidean space  $\mathbb{R}^N$ . We will show that if  $N > 2m + 1$ , then there is a nonzero vector  $a \in \mathbb{R}^N$  such that, if  $R^{N-1}$  is the subspace orthogonal to  $a$ , then the linear projection  $\pi : \mathbb{R}^N \rightarrow \mathbb{R}^{N-1}$  is a one-to-one immersion on  $M$ . Iterating this construction will complete step 1.

To begin, define a map  $h : M \times M \times \mathbb{R} \rightarrow \mathbb{R}^N$  by

$$h(x, y, t) = t(x - y).$$

Also define a map  $g : TM \rightarrow \mathbb{R}^N$  by

$$g(x, v) = v.$$

Since  $\dim(M \times M \times R) = 2m + 1 < N$  and also  $\dim(TM) = 2m < N$ , Sard's Theorem implies that there is a vector  $a \in \mathbb{R}^N$  which belongs neither to  $h(M \times M \times R)$  nor to  $g(TM)$ .

Note that  $a \neq 0$ , since 0 lies in the images of both  $h$  and  $g$ .

Now let  $R^{N-1}$  be the orthogonal complement in  $\mathbb{R}^N$  of the nonzero vector  $a$ , and  $\pi : \mathbb{R}^N \rightarrow \mathbb{R}^{N-1}$  the orthogonal projection.

First note that the map  $\pi|_M$  is one-to-one. For suppose  $\pi(x) = \pi(y)$  for some pair of distinct points  $x, y \in M$ . Then  $x - y = ta$ , for some non-zero real number  $t$ . But then  $h(x, y, 1/t) = a$ , contradicting the choice of  $a$ .

Similarly, we note that the map  $\pi|_M$  must be an immersion. For suppose  $v$  is a nonzero tangent vector in  $T_x M$  such that  $d\pi_x(v) = 0$ . Since  $\pi : \mathbb{R}^N \rightarrow \mathbb{R}^{N-1}$  is a linear map, we can write

$$d\pi_x(v) = \pi(v) = 0.$$

But this means that  $v = ta$ , for some  $t \neq 0$ , and hence that

$$a = g(x, (1/t)v),$$

which again contradicts the choice of  $a$ .

Thus  $\pi|_M : M \rightarrow \mathbb{R}^{N-1}$  is a one-to-one immersion. If we still have  $N - 1 > 2m + 1$ , rename  $\pi(M) \subset \mathbb{R}^{N-1}$  as simply  $M$  and repeat the above argument.

Iterating the argument until the dimension of the ambient Euclidean space has dropped to  $2m + 1$ , we end with a one-to-one immersion of  $M$  into  $\mathbb{R}^{2m+1}$ , completing Step 1 of the proof.

**Step 2.** In this section, we will use a partition of unity to help us modify the one-to-one immersion  $f : M^m \rightarrow \mathbb{R}^{2m+1}$  in order to make it proper. (Recall, this means that the inverse images under  $f$  of compact sets in  $\mathbb{R}^{2m+1}$  are compact sets in  $M$ .)

## Lemma

*On any smooth manifold  $M$  there exists a smooth proper map  $\phi : M \rightarrow \mathbb{R}$ .*

**Proof.** Let  $\{U_\alpha\}$  be a covering of  $M$  by open subsets which have compact closures, and  $\{\phi_n : n = 1, 2, 3, \dots\}$  a countable partition of unity subordinate to this covering.

Define the function  $\phi : M \rightarrow \mathbb{R}$  by

$$\phi(x) = \sum_{n=1}^{\infty} n\phi_n(x).$$

If  $\phi(x) \leq k$ , then at least one of the first  $k$  functions  $\phi_1, \dots, \phi_k$  must be nonzero at  $x$ .

Therefore

$$\phi^{-1}([-k, k]) \subset \bigcup_{n=1}^k \{x : \phi_n(x) \neq 0\},$$

which is a set with compact closure. Hence the function  $\phi$  is proper.

Using the above lemma, we now prove that every smooth  $m$ -dimensional manifold  $M^m \subset \mathbb{R}^N$  embeds in  $\mathbb{R}^{2m+1}$ . We begin, thanks to Step 1, with a one-to-one immersion  $f : M^m \rightarrow \mathbb{R}^{2m+1}$ . Compose  $f$  with a diffeomorphism of  $\mathbb{R}^{2m+1}$  to its own open unit ball  $B^{2m+1}$ , and let this be the new  $f$ . Now  $f : M^m \rightarrow B^{2m+1} \subset \mathbb{R}^{2m+1}$ .



Let  $\phi : M \rightarrow \mathbb{R}$  be a smooth proper map, as promised by the above lemma.

Combine  $f$  and  $\phi$  to get a new one-to-one immersion  $F : M^m \rightarrow \mathbb{R}^{2m+2}$  by defining

$$F(x) = (f(x), \phi(x)).$$

Then repeat Step 1 to find a suitable nonzero vector  $a$  in  $\mathbb{R}^{2m+2}$  so that if  $\mathbb{R}^{2m+1}$  is now the subspace orthogonal to  $a$ , and  $\pi : \mathbb{R}^{2m+2} \rightarrow \mathbb{R}^{2m+1}$  the orthogonal projection, then  $\pi \circ F : M^m \rightarrow \mathbb{R}^{2m+1}$  is a one-to-one immersion.

But since the set of suitable choices for  $a$  is, by Sard's Theorem, the complement of a set of measure zero, we can make sure that  $a$  does not lie on the  $x_{2m+2}$  axis that was added to the original  $\mathbb{R}^{2m+1}$  in defining  $F$ .

**Problem** Show that with this choice of  $a$ , the one-to-one immersion  $\pi \circ F : M^m \rightarrow \mathbb{R}^{2m+1}$  is proper, and hence, is an embedding.

This completes the proof of the Whitney Embedding Theorem for smooth manifolds already embedded in some high-dimensional Euclidean space.

## The Whitney Embedding Theorem.

**It remains to be shown:** Given an abstract smooth manifold  $M^m$ , use a smooth partition of unity on it to find a smooth embedding of  $M^m$  into some high-dimensional Euclidean space  $\mathbb{R}^N$ .

We will discuss this in class.

**Conclusion.** Abstract smooth manifolds are really no more abstract than the "concrete" smooth manifolds which appear as subsets of Euclidean spaces. Hence the Whitney Embedding Theorem applies equally to them, and embeds such manifolds of dimension  $m$  into  $\mathbb{R}^{2m+1}$ .

**Comment.** A more difficult theorem of Whitney shows that such manifolds actually embed into  $\mathbb{R}^{2m}$ .